Informe

Sundman's series and related problems

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Summary. One of the topics of the international competition opened in 1885 for the prize of Sweden's King Oscar II was, as it is well known, to find the solution of the n-body problem of the celestial mechanics by means of power series uniformly convergent for all values of t. Later works and difficulties fastened the idea that the solution of this problem for $n \geq 3$ could not be attained without the introduction of new and complicated trascendental functions.

However, in 1912, Karl Sundman, Head of Helsingfors's Observatory for many years, gave the exact solution of the three-body problem, not certainly in the sense of that of Euler, Lagrange and others, but in the sense of the theory of analytical functions, with the only tool of known theorems of mathematical analysis and without introducing any new special function.

The main results due to Sundman are:

i) In order that a simultaneous collision takes place it is necessary that the angular momentum vector be null.

ii) In the case of binary collisions the independent variable $\tau = \int_0^t \frac{dt}{r}$ where $r = \mathrm{Min}\,(r_1 r_2 r_3)$, r_x mutual distances, the integral being convergent, regularizes the equations of the motion.

iii) The fundamental theorem in virtue of which: if the angular momentum vector $C \neq 0$, the coordinates and the time can be developed into power series in the regularization variable τ -pseudotime- convergent in the infinite strip of the complex plane $u = \tau + i\theta$ defined by $-\infty < \tau < \infty$ and $|\theta| < \delta$ where δ depends only on the masses and the initial conditions, corresponding to the real-axis $-\infty < \tau < \infty$ of the u-plane the real-axis $-\infty < t < \infty$

Unfortunately for the later development of these ideas, as we shall see, Sundman has made finally a conformal mapping, by means of Poincaré's transformation of the strip mentioned above, of the u-plane into the unit circle of another w-plane. By means of this transformation he has expressed the solution of that problem in power series in w convergent in the unit circle $|w| < 1$ corresponding at the interval $|R(w)| < 1$ the real-axis $-\infty < t < \infty$

What are the reasons that these so beautiful and important results of Sundman, in which researches the greater astronomers and mathematiciens failed during two centuries, have only been known in small circles...nur in kleinem Kreise bekant... as Siegel has said in his beautiful book of the yellow series, in spite of the simplifications due to Levi-Civita, Hadamard, Birkoff...

This question is connected with this other: Is it true the recent sentence of Chazy (1952), based on calculations of Belorizky we shall soon mention, and which expresses the old views held by most astronomers, saying that "Mieux vaut en pratique employer les séries divergentes anterieures que les séries convergentes de Sundman"

In a series of papers appeared since 1931 D.Belorizky has studied the convergence of the solution of the three-body problem given by Sundman arriving to the conclusion that Sundman's series are very slowly convergent and therefore useless for all practical purposes. The arguments of Belorizky are not, however, very convincing for: 1st. Instead of analysing the convergence of Sundman's series in power in τ, Belorizky has studied the series obtained by application of Poincaré's transformation, showing really that this transformation

is inadequate for the numerical solution of the three-body problem.
2nd. Since in the example used by Belorizky, a special Lagrange's equilateral solution, a binary collision cannot occur, the use of a regularization variable is unnecessary. Belorizky himself has found that in this case one has $\tau = At$ (A, a constant) and that the solution in power series in τ are as rapidly convergent as the known expansions

$$\text{sen } t = t - \frac{t^3}{3!} + \ldots \quad \cos t = 1 - \frac{t^2}{2!} + \ldots$$

This same observation can be made to the recent example elaborated by Vernić with the aim of contradicting Belorizky's conclusions. This example, early calculated by Zumkley for an interval $0 \leq t \leq 10$ (Gauss's const. = 1; sidereal year = 2π) has allowed Vernić to obtain Sundman's series enough convergent because their first coefficients decrease very rapidly.

But as there is no possibility of binary collisions for $t \geq 0$, at the same results and more simply one arrives without the use of regularization variable, by direct application of Taylor's series, as I myself have calculated.

In this note I make another application of Sundman's series, but choosing this time an example in which there must occur binary collisions, and therefore Taylor's series cannot be applied directly.

By a theorem of Fransen and Wilczynski the only isosceles solutions of the three-body problem are:

i) The motion, not rectilinear, with a fixed axis of symmetry;
ii) The motion, not planar, with a fixed plane of symmetry; and
iii) The motion on a plane with a fixed axis of simmetry,

being equal, in all three cases, the masses placed on the base of the isosceles triangle.

In the 3rd case, as Wintner has observed in his text-book "The analytical Foundation of Celestial Mechanics", a simultaneous collision, but not a binary collision, can be excluded by suitable choic of the initial conditions. Since in this case the angular momentum vector vanishes, Sundman's theorem cannot be applied without excluding the possibility of such a simultaneous collision of the three bodies.

It is shown that if $m_o = 1 - 2\mu = 0.8$; $m_1 = m_2 = \mu = 0.1$ and for $t = t_o = 0$: $x^o = 8$, $y^o = 1$, $\dot{x}^o = 1$; $\dot{y}^o = -8$ where x,y denotes the heliocentric coordinates of m_1 with m_o as Sun, then:

i) No simultaneous collisions can occur;

ii) The pseudotime $\tau = \int_0^t dt/2y$ regularizes the equations of motion, the integral being convergent for all values of t.

iii) Sundman's series are in this case more rapidly convergent than the known expansion of the exponential function

$$e^{-16\tau} = 1 - 16\tau + \frac{1}{2!}(16\tau)^2 - \dots$$

and, by means of them, we can determine both the collision path and the abscissa and date of the binary collision, taking into account no much more than 35 terms, with a very good approximation and a trivial bound for the error.

As a control one can use the energy integral as well as the integral obtained by successive approximations. After finding the abscissa and the pseudotime τ_1 of the binary collision, new expansions into power series in $\tau - \tau_1$ still more rapidly convergent are obtained. (To be published in Anales de la Academia Nacional de Ciencias Exactas, Físicas y Naturales).